MATH4210: Financial Mathematics Tutorial 4

Jiazhi Kang

The Chinese University of Hong Kong

jzkang@math.cuhk.edu.hk

5 October, 2022

Review on Normal r.v.

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$$

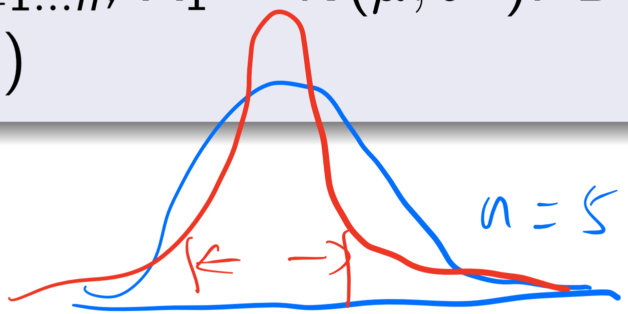
$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \quad (\mathbb{E} \text{ is linear})$$

Question

Assume a sequence of i.i.d. r.v.s $\{X_i\}_{i=1\ldots n}$, $X_1 \sim N(\mu, \sigma^2)$. *Denote by* $Y := \frac{1}{n}\sum_{i=1}^{n} X_i$. *Find* $\mathbb{E}[Y]$ *and* $Var(Y)$

Since X_i's are i.i.d.

So $\forall i \in [\![1, n]\!]$, $\mathbb{E}[X_i] = \mu$.



Hence. $\mathbb{E}[Y] = \frac{1}{n} \cdot n \cdot \mu = \mu$.

Recall $Var\left(\sum_{i=1}^{n} c_i Z_i\right) = \sum_{i=1}^{n} c_i^2 \, Var(Z_i) + \sum_{1 \le i \ne j \le n} c_i c_j \, Cov(Z_i, Z_j)$

$$Var(Y) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}\sum_{1 \le i \ne j \le n} \underbrace{Cov(X_i, X_j)}_{=0} = \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}.$$

Review on Normal r.v.

① $Y \sim N(\mu_Y, \sigma_Y^2) \iff \mathbb{E}[e^{i\theta Y}] = \underline{e^{i\theta\mu_Y - \frac{1}{2}\theta^2\sigma_Y^2}}$.

Fix $\theta \in \mathbb{R}$.

$$\mathbb{E}[e^{i\theta Y}] = \mathbb{E}[e^{i\theta \cdot \sum_{j=1}^{n} X_j}] = \mathbb{E}[\prod_{j=1}^{n} e^{i\theta X_j}]$$

Question

Assume a sequence of independent r.v.s $\{X_i\}_{i=1\dots n}$, for any $i \in [|1, n|]$, $X_i \sim N(\mu_i, \sigma_i^2)$. Denote by $Y := \sum_{i=1}^{n} X_i$. Show that Y is gaussian. Find the parameters of Y.

Since X_j's are independent.

Then: $\mathbb{E}[\prod_{j=1}^{n} e^{i\theta X_j}] = \prod_{j=1}^{n} \underline{\mathbb{E}[e^{i\theta X_j}]}$

$$= \prod_{j=1}^{n} e^{i\theta \cdot \mu_j - \frac{1}{2}\theta^2\sigma_j^2}$$

$$= e^{i\theta \underbrace{\sum_{j=1}^{n}\mu_j}_{\mu_Y} - \frac{1}{2}\theta^2 \underbrace{\sum_{j=1}^{n}\sigma_j^2}_{\sigma_Y^2}}$$

$Y \sim N(\sum_{j=1}^{n}\mu_j, \sum_{j=1}^{n}\sigma_j^2)$

$X \sim N(\mu, \sigma)$

$X = \mu + \sigma \cdot Z$.

$Z \sim N(0, 1)$

$\square$

Review on Normal r.v.

① X is not defined on $\mathbb{R}$

② Fix $x > 0$

$\mathbb{P}[X \le x] = \mathbb{P}[\ln X \le \ln x]$

Since $\ln X \sim N(\mu, \sigma^2)$

Then $\mathbb{P}[\ln X \le \ln x]$

$$= \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

Question

Assume X follows log-normal distribution with parameters μ, σ^2. Find the probability density function of X.

Recall that $X \sim LN(\mu, \sigma^2)$ if

$$\ln(X) \sim N(\mu, \sigma^2)$$

$$\frac{d}{dx} \mathbb{P}[\ln X \le \ln x] = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \cdot \frac{1}{x}$$

So $f_X(y) = \frac{1}{\sqrt{2\pi}\sigma y} \cdot e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}$ for $y > 0$

$\square$

Review on Normal r.v.

① $\mathbb{E}[X]=0 \quad \mathbb{E}[X^2]=1$

Fix $n\in\mathbb{N}^*$, $\mathbb{E}[X^n]=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^n\cdot e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^{n-1}\cdot \underline{x\cdot e^{-\frac{x^2}{2}}}dx$

Denote by $u(x)=x^{n-1}$, $v(x)=-e^{-\frac{x^2}{2}}$

Question

Assume $X\sim N(0,1)$. For $n\in\mathbb{N}^*$, find $\mathbb{E}[X^n]$.

Remark

If $Y\sim N(\mu,\sigma^2)$, we can write $\mathbb{E}[Y^n]$ explicitly by considering

$$Y=\mu+\sigma Z,\quad Z\sim N(0,1).$$

$v'(x)=x\cdot e^{-x^2/2}$

Then, by IBP

$$\mathbb{E}[X^n]=\frac{1}{\sqrt{2\pi}}\left(\left[-x^{n-1}\cdot e^{-\frac{x^2}{2}}\right]_{-\infty}^{+\infty}-\int_{\mathbb{R}}(n-1)x^{n-2}\cdot(-e^{-\frac{x^2}{2}})dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(0+(n-1)\cdot\int_{\mathbb{R}}x^{n-2}e^{-x^2/2}dx\right)=(n-1)\mathbb{E}[X^{n-2}]$$

$\mathbb{E}[X] = 0.$ $\mathbb{E}[X^2] = 1.$

$\forall$ n odd, $\mathbb{E}[X^n] = 0$

Answer:

$$\mathbb{E}[X^n] = \begin{cases} 0 & \text{if } n = 2k-1 \text{ for some } k \\ \frac{(2k)!}{2^k \cdot k!} & \text{if } n = 2k. \text{ for some } k. \end{cases}$$

$$\mathbb{E}[X^{2k-1}] = 0.$$

$$\mathbb{E}[X^{2k}] = \frac{(2k)!}{2^k\, k!}$$

Convergence of r.v.s

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

Convergen a.e. in real analysis

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

$= \mathbb{P}[\{\lim_{n\to+\infty} X_n = X\}] = 1$

$$\mathbb{P}[\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1$$

Converge. in measure

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

$$\lim_{n\to\infty} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

Convergence of r.v.s

$X_n \xrightarrow{a.s.} X$ if $\mathbb{P}(\{\lim X_n = X\}) = 1$

$X_n \xrightarrow{\mathbb{P}} X$. if $\forall \rho > 0$,

$\lim_{n \to +\infty} \mathbb{P}(\{|X_n - X| \geq \rho\}) = 0$

Question

Let X and $\{X_n\}$ be $\mathbb{R}$ valued (sequence of) r.v.s. Assume $X_n \to X$ a.s.. Show that $X_n \to X$ in probability.

Proof: Since $X_n \to X$ a.s.

we have $\mathbb{P}(\{\lim_{n \to +\infty} X_n = X\}) = 0$

Fix $\rho > 0$. and $n \in \mathbb{N}$.

$A_n = \bigcup_{m \geq n} \{|X_m - X| \geq \rho\}$

$\{|X_n - X| \geq \rho\} \subseteq A_n$.

$\mathbb{P}(\{|X_n - X| \geq \rho\}) \leq \mathbb{P}(A_n)$. --- $(*)$

If we take limit equal to 0, then ---

So we study $\mathbb{P}(A_n)$.

$\{A_n\}_{n \geq 1}$ is a decreasing sequence.

$\mathbb{P}(A_1) \leq \mathbb{P}(\Omega) = 1$.

Then $\mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_n\right) = \lim_{n \to +\infty} \mathbb{P}(A_n)$ —— $(**)$

Consider $B = \{|\lim_{n \to +\infty} X_n - X| = 0\}$. $\mathbb{P}(B) = 1$

Fix $\omega \in B$.

Since $X_n \to X$ a.s., there exists $N > 0$

$|X_n(\omega) - X(\omega)| < \varepsilon$ for all $n \geq N$

But $\forall n \geq N$, $\omega \notin A_n \Rightarrow B \cap \left(\bigcap_{n=1}^{+\infty} A_n\right) = \phi$

So $\mathbb{P}\left(B \cup \left(\bigcap_{n=1}^{+\infty} A_n\right)\right) \leq \mathbb{P}(\Omega) = 1$

$$\| $$

$$\mathbb{P}(B) + \mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_n\right)$$

$\Rightarrow \mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_n\right) + 1 \leq 1$

$\Rightarrow \mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_n\right) = 0$ --- $(***)$

By ( ) ( ), ( )

$$\lim_{n \to +\infty} \mathbb{P}(|X_n - X| \geq \rho) \leq \lim_{n \to +\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_n\right) = 0.$$

So we conclude: $X_n \to X$ in probability.